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A Study of The Exact Solutions and Conservation Laws of The Classical Lonngren Wave Equation for Communication Signals

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Abstract

This study undertakes a comprehensive examination of the classical Lonngren wave equation, a fundamental computational model used for simulating electrical signals in semiconductor materials, with specific emphasis on the tunnel diode. The primary objective of this study is to attain novel and more comprehensive solutions beyond those documented in existing literature. To achieve this goal, we have employed well-established mathematical methods, specifically analysis via Lie symmetry, coupled with other specialized techniques such as the power series method and Jacobi elliptic expansion technique. Notably, this marks the inaugural application of these methodologies to the classical Lonngren wave equation, signifying a pioneering endeavor in the exploration of this equation using these analytical tools. These methodologies yield solutions characterized by elliptic functions. The results are visually presented through 3D, 2D, and density plots, effectively illustrating the characteristics of these solutions. The visual representations reveal a range of patterns, including periodic and singular periodic solutions. Furthermore, the paper applies the multiplier method and leverages the conservation theorem introduced by Ibragimov to derive conserved vectors. These conserved vectors play a pivotal role in the examination of physical quantities, such as energy and momentum conservation, thereby enhancing our understanding of the underlying physics within the system.

Keywords: Lie symmetry analysis; group invariant solutions; conserved vectors; multiplier method; Ibragimov's theorem.

1 Introduction

Nonlinear partial differential equations (NLPDEQs) are of paramount importance due to their wide-ranging applications in diverse scientific and engineering disciplines. The study of nonlinear phenomena stands as a crucial and prominent area in contemporary science. A multitude of fields, including plasma physics, string theory, optical fibers, solid-state physics, biomechanics, computational physics, and gas dynamics, prominently feature physical phenomena primarily governed by nonlinear equations. As a consequence, extensive research efforts continue to be devoted to the study of NLPDEQs.

Properly analyzing nonlinear physical processes necessitates the investigation of exact solutions to NLPEQs. It is essential to acknowledge that no singular approach is universally applicable to solving all NLPEQs, given their inherent complexity. Consequently, a diverse array of techniques has been developed to effectively tackle these equations. Some of the established techniques include the extended homoclinic-test approach [5], the generalized Riccati equation mapping method [31], the tanh-coth technique [24], the Bäcklund transformation approach [8], the Painlevé expansion approach [4], the Adomian decomposition method [22], the extended simplest equation method [15], the Hirota technique [9], Lie symmetry analysis [20, 19], the bifurcation approach [29], Kudryashov's technique [14], the tanh-function method [23], and the rational expansion method [28], among others.

These techniques play a crucial role in unravelling the intricacies of nonlinear phenomena, providing invaluable insights, and paving the way for devising effective strategies to address real-world challenges. The application of such methodologies significantly enhances our understanding of complex physical systems governed by NLPEQs and contributes to advancements in various scientific and engineering disciplines.

Conservation laws play a pivotal role in both the identification of solutions and the reduction of partial differential equations (PDEQs). They serve as essential tools for exploring integrability and linearization mappings, establishing the existence and uniqueness of solutions, and analyzing the stability and global behavior of solutions [11]. Various techniques have been developed for the construction of conservation laws for differential equations, including the characteristic method, multiplier approach [19], Noether's theorem [18], symmetry-based methods and Ibragimov method [10], among others. The application of these techniques has been evident in numerous papers dedicated to the pursuit of conservation laws for diverse PDEQs.

In the year 1975, Lonngren introduced what has come to be known as the classical Lonngren wave equation (CLWE) in reference [17], and it is expressed as follows:

$$(u_{xx} - \alpha u + \beta u^2)_{tt} + u_{xx} = 0. \tag{1}$$

In the provided equation (1), α and β are defined as arbitrary constants. The temporal and spatial variables are denoted by t and x, respectively, with the wave function u being a function of these components [30]. Furthermore, the constant β serves as a coefficient that characterizes the nonlinearity inherent in equation (1). It is noteworthy that the derivation of this equation finds its physical basis in the behavior of electrical signals within Sony's tunnel diode, which falls under the category of semiconductor materials [17]. Additionally, equation (1) finds utility in elucidating the propagation of electrical signals through materials exhibiting semiconductor properties, as well as the mechanisms underlying energy storage in circuits featuring electric charge [3].

In the past decade, numerous authors have extensively investigated the suggested CLWE. Zhang et al. [30] found rational, trigonometric, and hyperbolic type travelling wave solutions by using the advanced $\exp(-\Psi(\eta))-$expansion function technique. Baskonus et al. [3] effectively employed equation (1) to expound upon the behaviour of electric signals within telegraph lines, employing the tunnel diode as a foundational element, and successfully derived its solution using the Sine-Gordon approach. Similarly, in reference [2], Akçaği and Aydemir discovered a travelling wave solution for equation (1) through the utilization of the $(G'/G)-$expansion function method and the modified tanh function technique. Furthermore, Yokuş [27] employed the auxiliary function technique to deduce a soliton solution for the Lonngren wave equation.

Motivated by the literature mentioned above, this research aims to find new exact solutions of the CLWE. To achieve the primary objective of the study, three distinct and efficient techniques will be employed: the Lie symmetry technique, the power series method and the extended Jacobi elliptic function technique. These techniques will be utilized to generate a multitude of exact solutions for the CLWE (1). It is noteworthy that the selection of methodologies in this study was deliberate, considering that none of these methods had been previously employed in the investigation of the CLWE. This strategic choice is particularly significant within our field, as it contributes to the expansion of methodological approaches applied to the CLWE, thereby enriching the overall understanding of the equation's behavior and characteristics. Additionally, numerical simulations will be conducted, employing three-dimensional and density plots, to demonstrate the dynamical behavior of solitary wave profiles for selected soliton solutions.

The research will focus on investigating the evolutionary dynamics and providing physical interpretations of the obtained exact soliton solutions, which are highly intriguing and offer valuable insights for further physical propositions. Notably, many of the derived exact soliton solutions are original and have not been previously reported in the existing literature.

Lastly, the multiplier method and Ibragimov's theorem will be applied to establish the conservation laws associated with the underlying model, contributing to a deeper understanding of the system's dynamics and its underlying physical implications. The study aims to make a significant contribution to the understanding of the CLWE and its soliton solutions, providing new insights into the behavior of this important class of nonlinear partial differential equations.

2 Lie Symmetry Analysis of (1)

Let us contemplate a one-parameter Lie group of infinitesimal transformations, characterized by the parameter k, of the CLWE;

$$\begin{aligned} \bar{t} &\longrightarrow t + k\,\tau(t,x,u) + O(k^2), \\ \bar{x} &\longrightarrow x + k\,\xi(t,x,u) + O(k^2), \\ \bar{u} &\longrightarrow u + k\,\eta(t,x,u) + O(k^2). \end{aligned}$$

The associated vector field Γ corresponding to the aforementioned transformation of the CLWE (1) can be expressed as follows,

$$\Gamma = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2}$$

The vector field Γ gives rise to Lie point symmetries, and it is imperative that vector Γ adheres to the invariant condition of the CLWE (1), namely,

$$\Gamma^{[4]}E\Big|_{(1)} = 0, \tag{3}$$

where,

$$E = (u_{xx} - \alpha u + \beta u^2)_{tt} + u_{xx}.$$

Here, $\Gamma^{[4]}$ is the fourth prolongation of Γ defined by,

$$\Gamma^{[4]} = X + \zeta_x \frac{\partial}{\partial u_x} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{ttxx} \frac{\partial}{\partial u_{ttxx}}.$$

Expanding (3) and splitting on various derivatives of u, we gain the system of PDEQs,

$$\tau_t = 0, \quad \tau_x = 0, \quad \tau_u = 0, \quad \xi_t = 0, \quad \xi_{xx} = 0, \quad \xi_u = 0, \quad \beta\eta - (\alpha - 2\beta u)\xi_x = 0.$$

Solving the above system we get the infinitesimals as,

$$\tau = C_1, \quad \xi = C_2 + C_3 x, \quad \eta = \frac{(\alpha - 2\beta u)}{\beta} C_3,$$

where, C_1, C_2 and C_3 are constants. Hence, the Lie point symmetries of the CLWE (1) are

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \beta x \frac{\partial}{\partial x} + (\alpha - 2\beta u) \frac{\partial}{\partial u}.$$

2.1 One-parameter groups of (1)

We now employ the Lie equations along with the initial conditions;

$$\begin{aligned}
\frac{d\bar{t}}{dk} &= \tau(\bar{t}, \bar{x}, \bar{u}), \qquad \bar{t}|_{k=0} = t, \\
\frac{d\bar{x}}{dk} &= \xi(\bar{t}, \bar{x}, \bar{u}), \qquad \bar{x}|_{k=0} = x, \\
\frac{d\bar{u}}{dk} &= \eta(\bar{t}, \bar{x}, \bar{u}), \qquad \bar{u}|_{k=0} = u,
\end{aligned}$$

to compute the one-parameter group of transformations. For each Γ_i, let G_{k_i} be the corresponding group. Using Lie equations, we get the following one-parameter groups for all the operators;

$$\begin{aligned}
&G_{k_1} : \bar{t} = t + k_1, \quad \bar{x} = x, \qquad \bar{u} = u, \\
&G_{k_2} : \bar{t} = t, \qquad \bar{x} = x + k_2, \quad \bar{u} = u, \\
&G_{k_3} : \bar{t} = t, \qquad \bar{x} = x e^{\beta k_3}, \quad \bar{u} = u e^{-2\beta k_3} + \frac{\alpha}{2\beta}\left(1 - e^{-2\beta k_3}\right).
\end{aligned}$$

Using the aforementioned groups, we may find the corresponding new solutions. Since each group G_{k_i} is a symmetry group, if $u = \Psi(t, x)$ is a solution of (1), then, the corresponding new solutions u_i are obtained as follows;

$$\begin{aligned}
u_1 &= \Psi(t - k, x), \\
u_2 &= \Psi(t, x - k), \\
u_3 &= \frac{\alpha}{2\beta}\left(1 - e^{-2\beta k}\right) + e^{-2\beta k}\Psi\left(t, x e^{-\beta k}\right).
\end{aligned}$$

2.2 Constructing group-invariant solutions of (1)

We now utilize the infinitesimal generators obtained in the previous subsection and perform symmetry reductions in order to obtain group invariant solutions for Equation (1).

Reduction 1. We consider the first Lie point symmetry $\Gamma_1 = \partial/\partial t$, by the characteristic method the corresponding Lagrange form is

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}.$$

We solve the above to obtain the invariants,

$$J_1 = x, \qquad J_2 = u,$$

and thus the group invariant solution $u = \omega(x)$. Substituting the value of u into (1) we obtain the second-order ordinary differential equation (ODEQ), $\omega'' = 0$, that is solved to obtain $\omega = C_1 x + C_2$, where C_1 and C_2 are constants of integration. Thus,

$$u(t,x) = C_1 x + C_2. \tag{4}$$

Reduction 2. Next, we consider the symmetry operator $\Gamma_2 = \partial/\partial x$. The characteristic equations for the operator Γ_2 are

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0},$$

which lead to the invariants,

$$J_1 = t, \qquad J_2 = u.$$

Thus,

$$u = \psi(t),$$

where ψ is an arbitrary function of t. Substituting this value of u into (1), we obtain the nonlinear ordinary differential equation (NLODEQ),

$$\alpha\psi'' - 2\beta\psi'^2 - 2\beta\psi\psi'' = 0.$$

Employing Maple for the resolution of the above equation we obtain the subsequent solutions,

$$\psi(t) = \pm\left(\frac{\sqrt{4C_1\beta t + 4C_2\beta + \alpha^2} + \alpha}{2\beta}\right),$$

where C_1 and C_2 are constants. Thus we have

$$u(t,x) = \pm\left(\frac{\sqrt{4C_1\beta t + 4C_2\beta + \alpha^2} + \alpha}{2\beta}\right). \tag{5}$$

Reduction 3. We now consider the Lie point symmetry $\Gamma_3 = \beta x\partial/\partial x + (\alpha - 2\beta u)\partial/\partial u$ that leads to the invariants,

$$J_1 = t, \qquad J_2 = ux^2 - \frac{\alpha x^2}{2\beta}.$$

The invariants imply that,

$$u = \frac{\alpha}{2\beta} + \frac{\phi(t)}{x^2},$$

where ϕ is an arbitrary function of t. Substituting this value of u into (1) we get the NLODEQ,

$$3\phi'' + \beta\phi'^2 + \beta\phi\phi'' + 3\phi = 0. \tag{6}$$

2.3 Solution of CLWE (1) via the power series method

In this section, we employ the power series solution method [16, 13] to address the NLODEQ (6). The utilization of power series methods has demonstrated their effectiveness in tackling intricate differential equations, spanning a broad spectrum from challenging semi-linear to intricate nonlinear ones. Therefore, the primary objective of this section is to derive the analytical solution for equation (1) by means of the power series technique, complemented by algebraic and differential transformations. Assume the formal solution to be of the form,

$$\phi(t) = \sum_{n=0}^{\infty} \rho_n t^n, \tag{7}$$

where $\rho_n (n = 0, 1, 2, \ldots)$ are constants to be determined. The derivatives of (7) to be used are given as,

$$\phi'(t) = \sum_{n=1}^{\infty} n\rho_n t^{n-1}, \qquad \phi''(t) = \sum_{n=2}^{\infty} n(n-1)\rho_n t^{n-2}. \tag{8}$$

By substituting the expressions derived from equations (7) and (8) into equation (6), we obtain the resulting equation,

$$\begin{aligned} 3\sum_{n=0}^{\infty}(n+1)(n+2)\rho_{n+1}t^n + 2\beta\left[\sum_{n=0}^{\infty}(n+1)\rho_{n+1}t^n\right]\left[\sum_{n=0}^{\infty}(n+1)\rho_{n+1}t^n\right] \\ + \beta\left[\sum_{n=0}^{\infty}\rho_n t^n\right]\left[\sum_{n=0}^{\infty}(n+1)(n+2)\rho_{n+2}t^n\right] = 0. \end{aligned} \tag{9}$$

By further simplifying equation (9) we obtain,

$$\begin{aligned} &6\rho_2 + 18\rho_3 t + 3\sum_{n=2}^{\infty}(n+1)(n+2)\rho_{n+1}t^n + 2\beta\rho_1^2 + 8\beta\rho_2^2 t \\ &+ 2\beta\sum_{n=2}^{\infty}\left[\sum_{k=0}^{n}(k+1)(n-k+1)\rho_{k+1}\rho_{n-k+1}t^n\right] + 2\beta\rho_0\rho_2 + 6\beta\rho_1\rho_3 t \\ &+ \beta\sum_{n=2}^{\infty}\left[\sum_{k=0}^{n}(n-k+1)(n-k+2)\rho_k\rho_{n-k+2}\right]t^n = 0. \end{aligned} \tag{10}$$

Comparing the coefficients of t in equation (10) we obtain,

$$3\rho_2 + \beta\rho_1^2 + \beta\rho_0\rho_2 = 0, \qquad \text{for } n = 0, \tag{11}$$

and

$$9\rho_3 + 4\beta\rho_2^2 + 3D\rho_1\rho_3 = 0, \qquad \text{for } n = 1. \tag{12}$$

Generally, for $n \geq 2$ the recursion relation in view of (10) is

$$\begin{aligned}\rho_{n+2} = -\frac{1}{3(n+1)(n+2)}\Bigg[&2\beta\sum_{k=0}^{n}(k+1)(n-k+1)\rho_{k+1}\rho_{n-k+1}\\ &+\beta\sum_{k=0}^{n}(n-k+1)(n-k+2)\rho_k\rho_{n-k+2}\Bigg].\end{aligned} \tag{13}$$

Utilizing equations (11), (12) and (13), it is possible to determine all the coefficients $\rho_n(n \geq 2)$ of the power series (7). For arbitrarily selected constant values ρ_0, ρ_1, ρ_2, and ρ_3, the remaining terms can also be determined in a unique manner through successive application of equations (11), (12) and (13). Now we endeavour to establish the convergence of the power series solution as delineated in equation (7). We introduce a new series,

$$\Omega = E(t) = \sum_{n=0}^{\infty} e_n t^n, \tag{14}$$

with $e_n = |\rho_n|\,(n = 0, 1, 2, 3)$ and

$$e_{n+2} = M\left[\sum_{k=0}^{n} e_{k+1}e_{n-k+1} + \sum_{k=0}^{n} e_k e_{n-k+2}\right], \qquad \text{for } n = 0, 1, 2, \ldots. \tag{15}$$

It is pertinent to note that from equation (13), without any loss of generality, we assert the following,

$$|\rho_{n+2}| \leq M\left[\sum_{k=0}^{n}|\rho_{k+1}|\,|\rho_{n-k+1}| + \sum_{k=0}^{n}|\rho_k|\,|\rho_{n-k+2}|\right], \qquad \text{for } n = 0, 1, 2, \ldots, \tag{16}$$

where,

$$M = \text{Max}\left\{\frac{2\,|\beta|}{3\,|(n+1)(n+2)|}, \frac{|\beta|}{3\,|(n+1)(n+2)|}\right\}.$$

Hence, through a comparative analysis of equation (16) and equation (13), it becomes apparent that,

$$|\rho_n| \leq e_n, \qquad n = 0, 1, 2, \ldots.$$

It can thus be stated that the series represented by (14) serves as a majorant series for the series denoted by equation (7). Subsequently, we aim to establish the existence of a positive radius of convergence for the series $\Omega = E(t)$. It is noteworthy to observe that through formal computations, one can attain the following result,

$$\begin{aligned}E(t) &= e_0 + e_1 t + e_2 t^2 + e_3 t^3 + \sum_{n=2}^{\infty} e_{n+2}t^{n+2}\\ &= e_0 + e_1 t + e_2 t^2 + e_3 t^3 + M\left[\sum_{n=2}^{\infty}\sum_{k=0}^{n} e_{k+1}e_{n-k+1}t^{n+2} + \sum_{n=2}^{\infty}\sum_{k=0}^{n} e_k e_{n-k+2}t^{n+2}\right]\\ &= e_0 + e_1 t + e_2 t^2 + e_3 t^3 + M\Bigg[\Big(e_0e_4 + 3e_1e_3 + 2e_2^2\Big)t^4 + \Big(e_0e_5 + 3e_1e_4 + 4e_2e_3\Big)t^5\\ &\quad + \Big(e_0e_6 + 3e_1e_5 + 4e_2e_4 + 2e_3^2\Big)t^6 + \Big(e_0e_7 + 3e_1e_6 + 4e_2e_5 + 4e_3e_4\Big)t^7\Bigg].\end{aligned}$$

In the following discussion, we consider the implicit functional equation presented as,

$$\begin{aligned}\mathcal{H}(t,\Omega) = \lambda - e_0 - e_1 t - e_2 t^2 - e_3 t^3 - M\Big[\Big(e_0 e_4 + 3e_1 e_3 + 2e_2^2\Big)t^4 \\ + \Big(e_0 e_5 + 3e_1 e_4 + 4e_2 e_3\Big)t^5 + \Big(e_0 e_6 + 3e_1 e_5 + 4e_2 e_4 + 2e_3^2\Big)t^6 \\ + \Big(e_0 e_7 + 3e_1 e_6 + 4e_2 e_5 + 4e_3 e_4\Big)t^7\Big].\end{aligned} \tag{17}$$

It is sufficient to show that $\mathcal{H}(t,\Omega)$ is analytic in the neighbourhood of $(0, e_0)$. From equation (17), it is evident that,

$$\mathcal{H}(0, e_0) = 0, \qquad \mathcal{H}'_\Omega(0, e_0) = 1 \neq 0.$$

Utilizing the implicit function theorem, as documented in [21, 6], we can assert that $\Omega = E(t)$ constitutes an analytic function within the proximity of the point $(0, e_0)$. Consequently, this implies that the power series (7) converges within a region surrounding the point $(0, e_0)$ in the plane. This, in turn, concludes the proof.

Thus, the power series solution of equation (6) can be written as,

$$\begin{aligned}\phi(t) &= \rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 + \rho_4 + \sum_{n=2}^{\infty} \rho_{n+2} t^{n+2} \\ &= \rho_0 + \rho_1 t - \frac{2\beta\rho_1^2}{6+2\beta\rho_0}t^2 - \frac{4\beta\rho_2^2}{9+3\beta\rho_1}t^3 - \sum_{n=2}^{\infty}\frac{1}{3(n+1)(n+2)} \\ &\quad \times \left[2\beta\sum_{k=0}^{n}(k+1)(n-k+1)\rho_{k+1}\rho_{n-k+1} + \beta\sum_{k=0}^{n}(n-k+1)(n-k+2)\rho_k\rho_{n-k+2}\right]t^{n+2},\end{aligned}$$

where the extended power series of the CLWE (1) is

$$\begin{aligned}u(t,x) = \frac{\alpha}{2\beta} + \frac{1}{x^2}\Bigg[\rho_0 + \rho_1 t - \frac{2\beta\rho_1^2}{6+2\beta\rho_0}t^2 - \frac{4\beta\rho_2^2}{9+3\beta\rho_1}t^3 - \sum_{n=2}^{\infty}\frac{1}{3(n+1)(n+2)} \\ \times \left\{2\beta\sum_{k=0}^{n}(k+1)(n-k+1)\rho_{k+1}\rho_{n-k+1} + \beta\sum_{k=0}^{n}(n-k+1)(n-k+2)\rho_k\rho_{n-k+2}\right\}t^{n+2}\Bigg].\end{aligned}$$

In physical applications it will be written in the form,

$$u(t,x) = \frac{\alpha}{2\beta} + \frac{1}{x^2}\left\{\rho_0 + \rho_1 t - \frac{2\beta\rho_1^2}{6+2\beta\rho_0}t^2 - \frac{4\beta\rho_2^2}{9+3\beta\rho_1}t^3\right\} + \cdots. \tag{18}$$

The wave profile of solutions (18) can be seen in Figure 1.

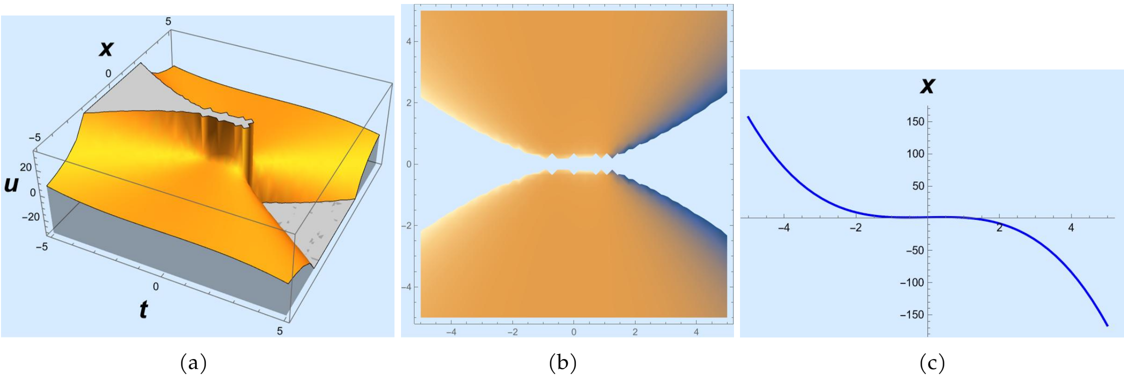


Figure 1: (a) 3D graph of singular solution (18) for $\alpha = 1, \beta = 2$, within the interval $-5 \leq t, x \leq 5$. (b) 2D density plot of solution (18). (c) 2D graph of solution (18) at $x = 1$ for the interval $-5 \leq t \leq 5$.

Reduction 4. Taking the linear combination of the symmetries Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 + c\Gamma_2$, by the characteristic method the corresponding Lagrange form is

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0}.$$

Solving the associated characteristic equations we gain the similarity variables and solution,

$$\xi = x - ct, \qquad u = \sigma(\xi).$$

These invariants reduce equation (1) into the NLODEQ,

$$2\beta c^2 \sigma\sigma'' + (1 - \alpha c^2)\sigma'' + 2\beta c^2 \sigma'^2 + c^2 \sigma'''' = 0. \tag{19}$$

2.4 Exact solution of CLWE (1) with the aid of extended Jacobi elliptic function expansion technique

This subsection is dedicated to the derivation of closed-form solutions for the equation represented by (1). The extended Jacobi elliptic function expansion technique, as elucidated in the works of [16, 12], is employed for this purpose.

Our objective is to ascertain solutions to the equation (1) in the context of the principal copolar trio of Jacobian elliptic functions, namely, the elliptic cosine $\text{cn}(\xi|\omega)$, elliptic sine $\text{sn}(\xi|\omega)$, and delta amplitude $\text{dn}(\xi|\omega)$, with the parameter ω constrained within the interval $0 \leq \omega \leq 1$. It is noteworthy that the remaining Jacobian functions can be expressed in terms of at least one of the elliptic functions within this copolar trio. For comprehensive details on these relationships, one can refer to the works of [7, 1]. Jacobi elliptic functions hold considerable significance due to their capacity to approximate trigonometric, hyperbolic, and, by extension, exponential functions, as summarized in Table 1.

Table 1: Copolar trio for $\omega = 0$ and $\omega = 1$.

	$\omega = 0$	$\omega = 1$
$\text{sn}(\xi\|\omega)$	$\sin\omega$	$\tanh\omega$
$\text{cn}(\xi\|\omega)$	$\cos\omega$	$\text{sech}\,\omega$
$\text{dn}(\xi\|\omega)$	1	$\text{sech}\,\omega$

We declare here that function $W(\xi)$ satisfies the first-order NLODEQs;

$$W'(\xi) + \Big\{\big(1 - W^2(\xi)\big)\big(1 - \omega + \omega W^2(\xi)\big)\Big\}^{1/2} = 0, \tag{20}$$

$$W'(\xi) - \Big\{\big(1 - W^2(\xi)\big)\big(1 - \omega W^2(\xi)\big)\Big\}^{1/2} = 0, \tag{21}$$

$$W'(\xi) + \Big\{\big(1 - W^2(\xi)\big)\big(\omega - 1 + W^2(\xi)\big)\Big\}^{1/2} = 0, \tag{22}$$

whose solutions are expressed in accordance with the Jacobi elliptic cosine, sine as well as the delta amplitude functions, respectively, as,

$$W(\xi) = \mathrm{cn}(\xi|\omega), \qquad W(\xi) = \mathrm{sn}(\xi|\omega), \qquad \text{and} \qquad W(\xi) = \mathrm{dn}(\xi|\omega). \tag{23}$$

Let, the fourth-order NLODEQ (19) be represented in the form,

$$\sigma(\xi) = \sum_{i=-m}^{m} A_i W(\xi)^i, \tag{24}$$

where, we aim to obtain the value of positive integer m by adopting the balancing procedure [25, 26], then we contemplate the subsequent solitary wave solutions.

2.4.1 Elliptic sine solution

It is worth noting that when applying the balancing procedure to equation (19), we determine that the parameter $m = 2$. Consequently, equation (24) can be expressed as,

$$\sigma(\xi) = A_{-2}W^{-2}(\xi) + A_{-1}W^{-1}(\xi) + A_0 + A_1W(\xi) + A_2W^2(\xi), \tag{25}$$

where A_{-2}, A_{-1}, A_0, A_1 and A_2 are constants to be determined. Utilizing the derived value of σ in equation (19) in conjunction with equation (21), we obtain a set of thirteen algebraic equations, viz.,

$$
\begin{aligned}
A_{-2}^2\beta c^2 + 6A_{-2}c^2 &= 0,\\
A_{-2}A_{-1}\beta c^2 + A_{-1}c^2 &= 0,\\
3A_{-1}^2\beta c^2 - 3A_{-2}\Big(c^2\big(\alpha + 20\omega + 20\big) - 1 - 2A_0\beta c^2\Big) - 8A_{-2}^2\beta c^2(\omega + 1) &= 0,\\
2A_{-2}A_1\beta c^2 - A_{-1}\Big(c^2\big(\alpha + 10\omega + 10\big) - 1 - 2A_0\beta c^2\Big) - 9A_{-2}A_{-1}\beta c^2(\omega + 1) &= 0,\\
A_{-2}\Big(c^2\big(\alpha\omega + \alpha + 4\omega^2 + 26\omega + 4\big) - \omega - 1 - 2A_0\beta c^2(\omega + 1)\Big) + 3A_{-2}^2\beta c^2\omega - A_{-1}^2\beta c^2(\omega + 1) &= 0,\\
A_{-1}\Big(c^2\big(\alpha\omega + \alpha + \omega^2 + 14\omega + 1\big) - \omega - 1 - 2A_0\beta c^2(\omega + 1)\Big) + 12A_{-2}A_{-1}\beta c^2\omega &\\
-2A_{-2}A_1\beta c^2(\omega + 1) &= 0,\\
A_2 - A_{-2}\omega\Big(c^2(\alpha + 4\omega + 4) - 1 - 2A_0\beta c^2\Big) - \alpha A_2c^2 + A_{-1}^2\beta c^2\omega + A_1^2\beta c^2 &\\
+2A_0A_2\beta c^2 - 4A_2c^2\omega - 4A_2c^2 &= 0,\\
\alpha A_1c^2\omega + \alpha A_1c^2 - 2A_0A_1\beta c^2\omega - 2A_{-1}A_2\beta c^2(\omega + 1) - 2A_0A_1\beta c^2 &\\
+12A_1A_2\beta c^2 + A_1c^2\omega^2 + 14A_1c^2\omega + A_1c^2 - A_1\omega - A_1 &= 0,
\end{aligned}
$$

$$A_2\Big(3A_2\beta c^2 + c^2\big(\alpha\omega + \alpha + 4\omega^2 + 26\omega + 4\big) - \omega - 1\Big) - A_1^2\beta c^2(\omega+1) - 2A_0A_2\beta c^2(\omega+1) = 0,$$
$$A_1\omega - \alpha A_1 c^2\omega + 2A_0A_1\beta c^2\omega + 2A_{-1}A_2\beta c^2\omega - 9A_1A_2\beta c^2(\omega+1) - 10A_1c^2\omega^2 - 10A_1c^2\omega = 0,$$
$$6A_0A_2\beta c^2\omega - A_2\Big(8A_2\beta c^2(\omega+1) + 3\omega\big(c^2(\alpha+20\omega+20)-1\big)\Big) + 3A_1^2\beta c^2\omega = 0,$$
$$A_1A_2\beta c^2\omega + A_1c^2\omega^2 = 0,$$
$$A_2c^2\omega\big(A_2\beta + 6\omega\big) = 0.$$

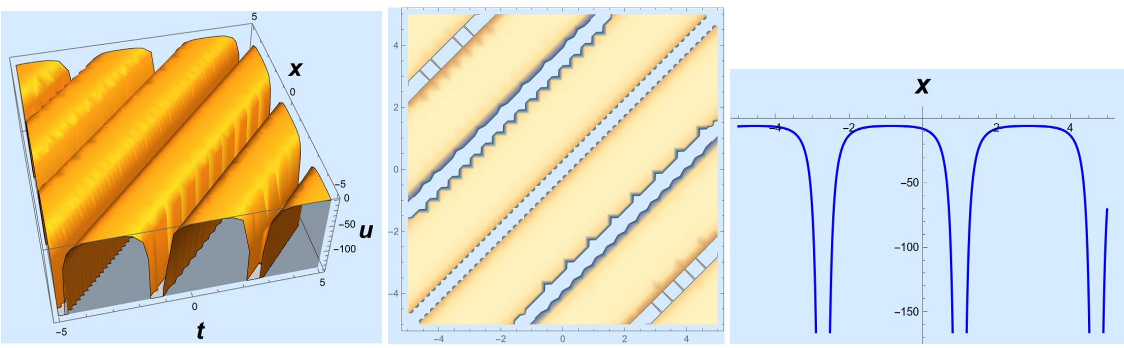
Employing Maple to solve the aforementioned system of equations, we obtain,

$$A_{-2} = -\frac{6}{\beta}, \quad A_{-1} = 0, \quad A_0 = \frac{c^2(\alpha+4\omega+4)-1}{2\beta c^2}, \quad A_1 = 0, \quad A_2 = -\frac{6\omega}{\beta}.$$

Consequently, the solution to (1) is

$$u(t,x) = \frac{\alpha c^2 + 4c^2\omega + 4c^2 - 1}{2\beta c^2} - \frac{6}{\beta}\,\mathrm{sn}^{-2}(x - ct \mid \omega) - \frac{6\omega}{\beta}\,\mathrm{sn}^2(x - ct \mid \omega). \tag{26}$$

The wave profile for the solution (26) can be seen in Figure 2.



(a) (b) (c)

Figure 2: (a) 3D graph of singular periodic soliton solution (26) for $\alpha = 1, \beta = 2, c = 1, \omega = 0.5$ within the interval $-5 \le t, x \le 5$. (b) 2D density plot of solution (26). (c) 2D graph of solution (26) at $x = 1$, for interval $-5 \le t \le 5$.

2.4.2 Elliptic cosine solution

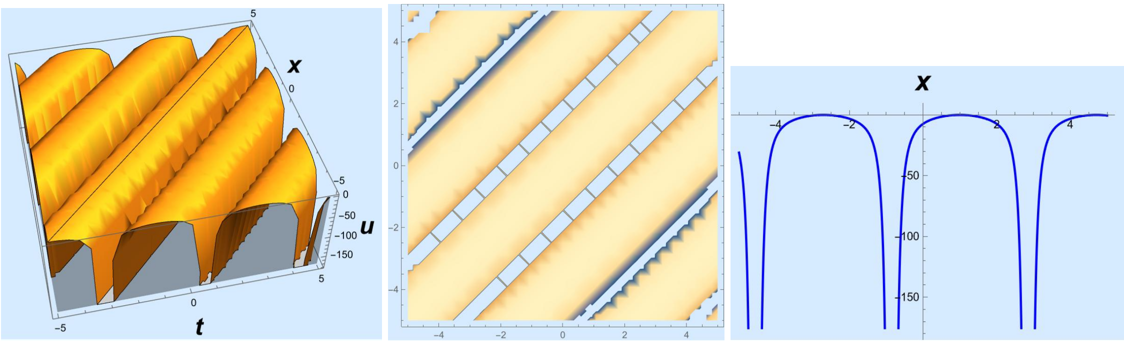
The solutions expressed in terms of elliptic cosine functions can be derived by substituting equations (25) into equation (19) and invoking NLODEQ (20). As before, the resultant parameter values in this case are

$$A_{-2} = \frac{6(\omega-1)}{\beta}, \quad A_{-1} = 0, \quad A_0 = \frac{c^2(\alpha-8\omega+4)-1}{2\beta c^2}, \quad A_1 = 0, \quad A_2 = \frac{6\omega}{\beta}.$$

Thus the solution to equation (1) is

$$u(t,x) = \frac{\alpha c^2 - 8c^2\omega + 4c^2 - 1}{2\beta c^2} + \frac{6(\omega-1)}{\beta}\,\mathrm{cn}^{-2}(x - ct \mid \omega) + \frac{6\omega}{\beta}\,\mathrm{cn}^2(x - ct \mid \omega). \tag{27}$$

The wave profile of solution (27) can be seen in Figure 3.



(a) (b) (c)

Figure 3: (a) 3D graph of singular periodic soliton solution (27) for $\alpha = 1, \beta = 2, c = 1, \omega = 0.5$ within the interval $-5 \leq t, x \leq 5$. (b) 2D density plot of solution (27). (c) 2D graph of solution (27) at $x = 1$ within the interval $-5 \leq t \leq 5$.

2.4.3 Elliptic delta solution

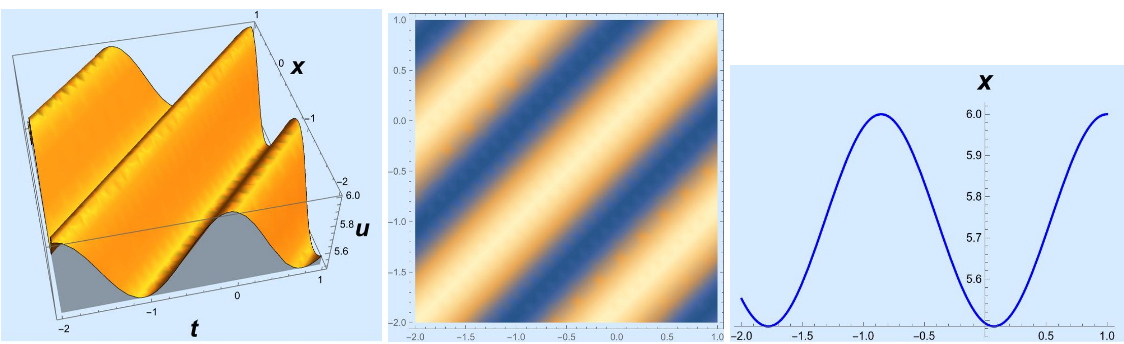
Likewise, solutions expressed in terms of elliptic delta functions can be obtained by substituting equations (25) into equation (19) and using NLODEQ (22). The resulting parameter values are

$$A_{-2} = \frac{6(1-\omega)}{\beta}, \qquad A_{-1} = 0, \qquad A_0 = \frac{c^2(\alpha + 4\omega - 8) - 1}{2\beta c^2}, \qquad A_1 = 0, \qquad A_2 = \frac{6}{\beta}.$$

Therefore, the solution to equation (1) is

$$u(t,x) = \frac{\alpha c^2 + 4c^2\omega - 8c^2 - 1}{2\beta c^2} + \frac{6(1-\omega)}{\beta}\,\mathrm{dn}^{-2}(x - ct \mid \omega) + \frac{6}{\beta}\,\mathrm{dn}^2(x - ct \mid \omega). \tag{28}$$

The wave profile of the solution described in equation (28) is depicted in Figure 4.



(a) (b) (c)

Figure 4: (a) 3D graph of periodic soliton solution (28) for $\alpha = 1, \beta = 2, c = 1, \omega = 0.5$ within the interval $-2 \leq t, x \leq 1$. (b) 2D density plot of solution (28). (c) 2D graph of solution (28) at $x = 1$ within the interval $-2 \leq t \leq 1$.

3 Conservation Laws for the CLWE

We now formulate conservation laws for the CLWE (1). In order to accomplish this objective, we employ two distinct techniques: the multiplier method as detailed in [19] and Ibragimov's theorem [10].

3.1 Conservation laws using the multiplier approach

The multiplier approach is highly regarded for its proficiency in addressing conserved quantities of differential equations, regardless of whether they exhibit variational principles. It is noteworthy that even when employing higher-order multipliers, the resultant outcomes remain consistent. Therefore, we shall focus on the zeroth-order multiplier denoted as $\Lambda = \Lambda(t, x, u)$ and leverage these multipliers to establish the conservation laws governing (1). It is essential to recall that the zeroth-order multipliers, denoted as Λ, are determined based on the determining equation,

$$\frac{\delta}{\delta u}\{\Lambda E\} = 0. \tag{29}$$

Expansion of (29) and following the steps similar to the algorithm of computing Lie symmetries, we obtain twenty three determining equations which upon simplification gives,

$$\Lambda_u = 0, \qquad \Lambda_{tt} = 0, \qquad \Lambda_{xx} = 0.$$

Solving the above system we get,

$$\Lambda = (C_1 x + C_2)t + C_3 x + C_4, \tag{30}$$

where C_1, C_2 and C_3 are constants. The required conserved quantities can then be found via the divergence identity given as,

$$D_t T^t + D_x T^x = \Lambda\left\{(u_{xx} - \alpha u + \beta u^2)_{tt} + u_{xx}\right\}, \tag{31}$$

where T^t is the conserved density and T^x is the spatial flux [19].

Thus, after some tedious computations the conserved vectors corresponding to the four multipliers are given below.

Case 1. For the first multiplier $\Lambda_1 = tx$, the corresponding conserved vector is given by,

$$\begin{aligned}
T_1^t &= \beta txuu_t - \beta xu^2 - \alpha txu_t + \alpha u_x + \frac{1}{2}txu_{txx} + \frac{1}{3}u_x - \frac{1}{6}xu_{xx} - \frac{1}{3}tu_{tx},\\
T_1^x &= txu_x - tu + \frac{1}{3}u_t - \frac{1}{3}xu_{tx} - \frac{1}{6}tu_{tt} + \frac{1}{2}txu_{ttx}.
\end{aligned}$$

Case 2. The second multiplier $\Lambda_2 = t$ yields the associated conserved vector as,

$$\begin{aligned}
T_2^t &= \alpha u - \frac{1}{6}u_{xx} - \alpha tu_t - \beta u^2 + 2\beta tuu_t + \frac{1}{2}tu_{txx},\\
T_2^x &= tu_x - \frac{1}{3}tu_{tx} + \frac{1}{2}tu_{ttx}.
\end{aligned}$$

Case 3. For the third multiplier $\Lambda_3 = x$, the corresponding conserved vector is expressed as,

$$\begin{aligned} T_3^t &= \frac{1}{2}xu_{txx} - \frac{1}{3}u_{tx} - \alpha xu_t + 2\beta xuu_t, \\ T_3^x &= \frac{1}{2}xu_{ttx} - u + xu_x - \frac{1}{6}u_{tt}. \end{aligned}$$

Case 4. For the final multiplier $\Lambda_4 = 1$, the corresponding conserved vector is given by,

$$\begin{aligned} T_4^t &= \frac{1}{2}u_{txx} + 2\beta uu_t - \alpha u_t, \\ T_4^x &= \frac{1}{2}u_{ttx} + u_x. \end{aligned}$$

3.2 Conservation laws using Ibragimov's theorem

The established methodology asserts that each infinitesimal symmetry provides a distinct conserved quantity, as elucidated in Ibragimov's work [10]. We employ Ibragimov's theorem to compute the conserved vectors pertaining to the CLWE (1). For a comprehensive grasp of this approach, interested readers are encouraged to refer to reference [10].

The adjoint equation for the CLWE (1) can be obtained from,

$$F^* \equiv \frac{\delta}{\delta u}\Big[v\left\{(u_{xx} - \alpha u + \beta u^2)_{tt} + u_{xx}\right\}\Big] = 0, \tag{32}$$

where $v = v(t, x)$, and this yields,

$$F^* \equiv v_{xx} - (\alpha - 2\beta u)v_{tt} + v_{ttxx} = 0, \tag{33}$$

which implies that the CLWE (1) is not self-adjoint. Consider equation (1) and the adjoint equation (33) as a system. The formal Lagrangian for this system is

$$\mathcal{L} = v\Big\{(u_{xx} - \alpha u + \beta u^2)_{tt} + u_{xx}\Big\}. \tag{34}$$

Recall that the CLWE (1) admits three infinitesimal symmetries. For each symmetry we derive the conserved vector (T^t, T^x) by using,

$$\begin{aligned} T^i = \xi^i\mathcal{L} &+ W\left[\frac{\partial\mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) - \cdots\right] \\ &+ D_j(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) + \cdots\right] + D_jD_k(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}} - \cdots\right], \end{aligned} \tag{35}$$

where,

$$W = \eta - \xi^j u_j.$$

Thus, the conserved vector (T^t, T^x) for each symmetry are given below:

Case 1. For $\Gamma_1 = \partial/\partial t$ the conserved vector is given by,

$$\begin{aligned} T_1^t &= 2\beta uu_tv_t - \alpha u_tv_t - \frac{1}{3}u_{tx}v_{tx} + \frac{1}{6}u_{txx}v_t + \frac{1}{2}u_tv_{txx} - \frac{1}{6}u_{tt}v_{xx} + \frac{1}{3}u_{ttx}v_x \\ &\quad + u_{xx}v + \frac{1}{2}u_{ttxx}v, \\ T_1^x &= u_tv_x + \frac{1}{2}u_tv_{ttx} - \frac{1}{3}u_{tt}v_{tx} - \frac{1}{6}u_{tx}v_{tt} + \frac{1}{3}u_{ttx}v_t + \frac{1}{6}u_{ttt}v_x - u_{tx}v - \frac{1}{2}u_{tttx}v. \end{aligned}$$

Case 2. For the symmetry $\Gamma_2 = \partial/\partial x$ the conserved vector from (35) is

$$\begin{aligned} T_2^t &= 2\beta u u_x v_t - \alpha u_x v_t + \frac{1}{6}u_{xxx}v_t - \frac{1}{6}u_{tx}v_{xx} - \frac{1}{3}u_{xx}v_{tx} + \frac{1}{3}u_{txx}v_x + \frac{1}{2}u_x v_{txx} \\ &\quad + \alpha u_{tx}v - 2\beta u_x u_t v - 2\beta u u_{tx} v - \frac{1}{2}u_{txxx}v, \\ T_2^x &= u_x v_x - \frac{1}{3}u_{tx}v_{tx} + \frac{1}{3}u_{txx}v_t - \frac{1}{6}u_{xx}v_{tt} + \frac{1}{6}u_{ttx}v_x + \frac{1}{2}u_x v_{ttx} - \alpha u_{tt}v + 2\beta u_t^2 v \\ &\quad + 2\beta u u_{tt} v + \frac{1}{2}u_{ttxx}v. \end{aligned}$$

Case 3. Finally for the symmetry $\Gamma_3 = \beta x\partial/\partial x + (\alpha - 2\beta u)\partial/\partial u$, we have

$$\begin{aligned} T_3^t &= 2\beta^2 x u u_x v_t - \alpha\beta x u_x v_t - \frac{1}{3}\beta u_t v_{xx} + \frac{2}{3}\beta u_{xx}v_t + \frac{1}{6}\beta x u_{xxx}v_t + \beta u_{tx}v_x \\ &\quad - \frac{1}{6}\beta x u_{tx}v_{xx} - \beta u_x v_{tx} - \frac{1}{3}\beta x u_{xx}v_{tx} + \frac{1}{3}\beta x u_{txx}v_x + \frac{1}{2}\beta x u_x v_{txx} \\ &\quad + 4\alpha\beta u_t v + \alpha\beta x u_{tx}v - 8\beta^2 u u_t v - 2\beta^2 x u_x u_t v - 2\beta^2 x u u_{tx} v - 2\beta u_{txx}v \\ &\quad - \frac{1}{2}\beta x u_{txxx}v - 4\alpha\beta u v_t + 4\beta^2 u^2 v_t + \beta u v_{txx} + \alpha^2 v_t - \frac{1}{2}\alpha v_{txx}, \\ T_3^x &= \beta x u_x v_x + \beta u_{tx}v_t - \frac{2}{3}\beta u_t v_{tx} - \frac{1}{3}\beta x u_{tx}v_{tx} + \frac{1}{3}\beta x u_{txx}v_t + \frac{1}{3}\beta u_{tt}v_x \\ &\quad - \frac{1}{2}\beta u_x v_{tt} - \frac{1}{6}\beta x u_{xx}v_{tt} + \frac{1}{6}\beta x u_{ttx}v_x + \frac{1}{2}\beta x u_x v_{ttx} - \alpha\beta x u_{tt}v \\ &\quad + 2\beta^2 x u_t^2 v + 2\beta^2 x u u_{tt} v - 3\beta u_x v - \frac{3}{2}\beta u_{ttx}v + \frac{1}{2}\beta x u_{ttxx}v \\ &\quad + 2\beta u v_x + \beta u v_{ttx} - \alpha v_x - \frac{1}{2}\alpha v_{ttx}. \end{aligned}$$

Remark 3.1. *The concept of conservation laws holds significant importance in the field of physics, encompassing both theoretical and quantum mechanics. In isolated physical systems, several fundamental quantities, including charge, mass, energy and momentum are conserved. These conserved quantities serve essential roles in various aspects of natural sciences, including integrability checks for differential equations. Furthermore, they play a crucial role in establishing the existence and uniqueness of solutions and in linearization mappings. Additionally, they are instrumental in analysing the global behaviour of solutions and stability.*

In this study, we utilized both the multiplier method and Ibragimov's approach, which enable the identification of conserved vectors for differential equations, whether or not they possess variational principles. It is worth noting that, while both techniques resulted in local conserved quantities, they yielded different numbers of such quantities, specifically four and three, respectively.

These conservation laws, as revealed in our research, hold significance in the realm of physical sciences. Physically speaking, the obtained conserved vectors signify the conservation of fundamental physical quantities such as energy and momentum. To be more precise, we observe that time translation symmetry is associated with energy conservation, whereas spatial translation symmetry is linked to the conservation of momentum. Moreover, the presence of the variable v in the three conserved vectors obtained through Ibragimov's technique indicates that the considered differential equation possesses an infinite number of conserved vectors. Additionally, the fact that we have a multiplier value of $\Lambda_4 = 1$ signifies that the CLWE (1) is in a conserved form.

4 Conclusions

In conclusion, this study undertook a thorough investigation of the classical Lonngren wave equation, a pivotal computational model extensively employed for simulating electrical signals within semiconductor materials, with a specific focus on its application to tunnel diodes. Notably, this research marked a groundbreaking endeavour as it presented, for the first time, a comprehensive analysis of the classical Lonngren wave equation employing advanced techniques such as Lie symmetry analysis, the power series method, and the Jacobi elliptic expansion technique. As a result, we successfully determined the symmetries and executed similarity reductions corresponding to each identified symmetry. This process allowed us to systematically analyze and categorize the solutions, providing a structured and comprehensive understanding of the mathematical properties inherent in the system.

The outcomes of these analytical approaches have yielded solutions characterized by Jacobi elliptic functions. All solutions presented in this study have undergone verification using Maple. This rigorous validation process ensures the accuracy and reliability of the obtained solutions, reinforcing the credibility of our analytical findings. What distinguishes these findings is their novelty and these have not hitherto been documented in the existing literature. To enhance comprehension, the results have been effectively depicted through a variety of visual representations, including 3D, 2D, and density plots. These graphical illustrations vividly portray diverse patterns, including periodic and singular periodic solutions.

Furthermore, the study employed the multiplier method and harnessed the conservation theorem introduced by Ibragimov to derive conserved vectors. These conserved vectors serve as crucial tools for examining and understanding physical quantities, particularly the conservation of energy and momentum. In doing so, they contributed significantly to the elucidation of the underlying physics governing the system under investigation.

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Conflicts of Interest The authors declare no conflict of interest.

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